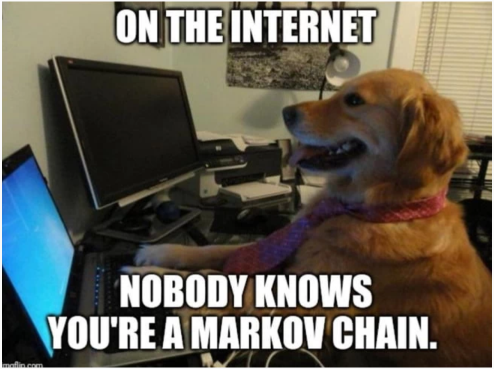
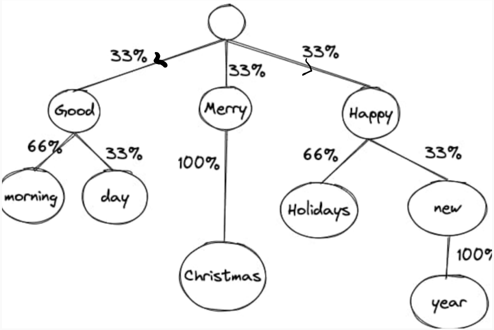
Games, graphs, and machines

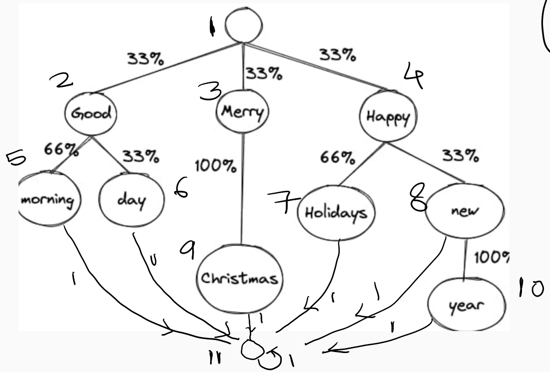


August 28, 2024

A text generator



$$A^4 = A^5 = A^6 = \cdots\cdots = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$



Let A be the corresponding transition matrix. The powers of A stabilise. When do they stabilise? What is the first row of A^{100}?

The Perron-Frobenius theorem

Theorem

Let A be the transition matrix of a Markov chain. Suppose there exists an n such that for every i and j, there is a path of length n from state i to state j. Then

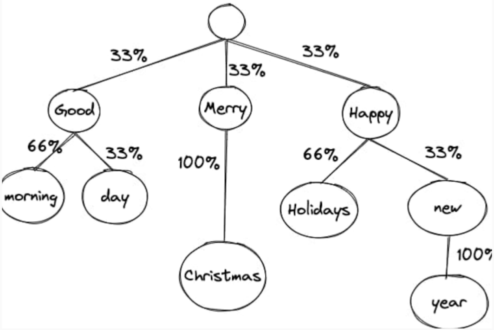
$A^{*n} =$ all 1's.

1. $\lim_{k\to\infty} A^k$ *exists.*
2. *The limiting matrix has identical rows, with non-negative entries summing to* 1*.*
3. *The limiting row vector v is the unique vector whose entries sum to* 1 *and which satisfies the equation*

$$vA = v.$$

Does it apply?

To the text generator? NO.



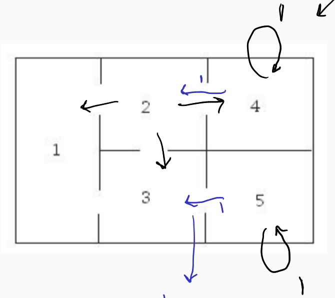
Does it apply?

To the random walk of the rats?

- with cheese/traps in rooms 4/5? No.
- without? Yes

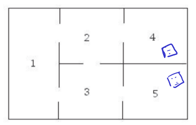
  ↳ $n=5$

Cheese



No cheese

With cheese/traps

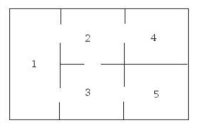



```
A = matrix([[0, 1/2, 1/2, 0, 0],
            [1/3, 0,1/3, 1/3, 0],
            [1/3, 1/3,0,0,1/3],
            [0,0,0,1,0],
            [0,0,0,0,1]])
(A^(100)).n(10)
```

≈ 0 ≈ 0

```
[1.2e-12 1.4e-12 1.4e-12    0.50    0.50]
[9.0e-13 1.0e-12 1.0e-12    0.62    0.38]
[9.0e-13 1.0e-12 1.0e-12    0.38    0.62]
[   0.00    0.00    0.00     1.0    0.00]
[   0.00    0.00    0.00    0.00     1.0]
```

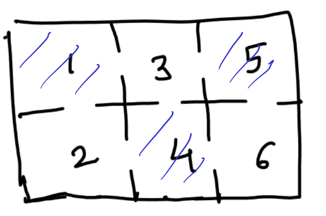
Without cheese/traps




```
A = matrix([[0, 1/2, 1/2, 0, 0],
            [1/3, 0,1/3, 1/3, 0],
            [1/3, 1/3,0,0,1/3],
            [0,1,0,0,0],
            [0,0,1,0,0]])
(A^(100)).n(10)

[0.20 0.30 0.30 0.10 0.10]
[0.20 0.30 0.30 0.10 0.10]
[0.20 0.30 0.30 0.10 0.10]
[0.20 0.30 0.30 0.10 0.10]
[0.20 0.30 0.30 0.10 0.10]
```

A slightly different maze

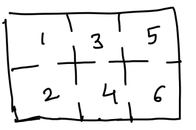
No n is both even & odd!



odd

even.

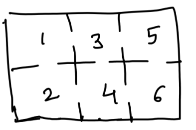
A slightly different maze




```
A = matrix([[0, 1/2, 1/2, 0, 0,0],
            [1/2,0,0,1/2,0,0],
            [1/3,0,0,1/3,1/3,0],
            [0,1/3,1/3,0,0,1/3],
            [0,0,1/2,0,0,1/2],
            [0,0,0,1/2,1/2,0]])
(A^(100)).n(10)
```

```
[0.29 0.00 0.00 0.43 0.29 0.00]
[0.00 0.29 0.43 0.00 0.00 0.29]
[0.00 0.29 0.43 0.00 0.00 0.29]
[0.29 0.00 0.00 0.43 0.29 0.00]
[0.29 0.00 0.00 0.43 0.29 0.00]
[0.00 0.29 0.43 0.00 0.00 0.29]
```

A slightly different maze

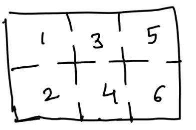



```
(A^(101)).n(10)
```

```
[0.00 0.29 0.43 0.00 0.00 0.29]
[0.29 0.00 0.00 0.43 0.29 0.00]
[0.29 0.00 0.00 0.43 0.29 0.00]
[0.00 0.29 0.43 0.00 0.00 0.29]
[0.00 0.29 0.43 0.00 0.00 0.29]
[0.29 0.00 0.00 0.43 0.29 0.00]
```

$\lim A^k$ does not exist!

A slightly different maze

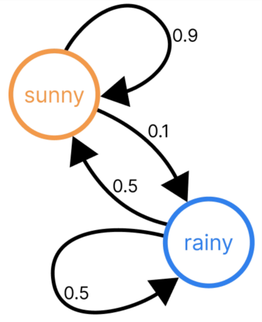



```
(A^(101)).n(10)

[0.00 0.29 0.43 0.00 0.00 0.29]
[0.29 0.00 0.00 0.43 0.29 0.00]
[0.29 0.00 0.00 0.43 0.29 0.00]
[0.00 0.29 0.43 0.00 0.00 0.29]
[0.00 0.29 0.43 0.00 0.00 0.29]
[0.29 0.00 0.00 0.43 0.29 0.00]
```

```
A.eigenvalues()

[1, 1/2, 1/6, -1/6, -1/2, -1]
```

Does it apply?

To the weather forecaster?



$$A = \begin{pmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{pmatrix}$$

$$(x \ y) \begin{pmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{pmatrix} = (x \ y)$$

and $x + y = 1$

$$0.9x + 0.5y = x$$
$$0.1x + 0.5y = y$$

$$0.9x + 0.5(1-x) = x$$

$$0.6x = 0.5$$
$$x = 5/6$$

Why does does Perron-Frobenius hold?

Key idea:

- only one eigenvalue is 1
- all the others have absolute value < 1